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The Motion of Shock Waves and Products of Detonation Confined between a Wall and a Rigid Piston

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In a previous paper, Aziz *et al.* [1] considered the motion of a rigid piston under the impact of a detonation wave. In their one-dimensional model the back-end of the "pipe" was open, allowing the products of detonation to escape. The present research considers the case when the detonation products cannot escape because the pipe is closed in the rear. The numerical results verify the conjecture that there is a critical value of the mass ratio, r, (ratio of the mass of the undetonated explosive to the piston mass) above which the trajectory of the piston is the same regardless of whether the pipe end is closed or open. A detailed examination of the piston motion indicates an apparent anomaly—the piston speed is increased above the "open-pipe" value before the shock wave rebounding off the closed end reaches the piston again. In order to explain this phenomenon a detailed analytical solution of the piston motion and flow field is carried out, for the case of polytropic detonation products $\gamma = 3$.

1. INTRODUCTION

The problem of the distribution of fluid properties behind a plane detonation wave in a condensed high explosive has been solved by G. I. Taylor [2]. The velocity imparted to a rigide piston by such a detonation head was calculated, for the case of polytropic combustion products with $\gamma = 3$, by Aziz, Hurwitz and Sternberg [1]. This was done for the case where there is a free back expansion into vacuum.

The present gas-dynamic problem differs form the one just described in that the detonation products are not allowed to expand backwards. Physically this corresponds to either having a rigid massive wall at the plane where the detonation starts, or, to the symmetric case of two detonation waves initiated at a plane. This is the one-dimensional analogy to the spherically and cylindrically symmetric cases which were treated elsewhere [8]. The advantage of the one dimensional model is that it allows an analytic solution to the problem and thus enhances our understanding of it. Figures 1 and 2 indicate the two situations some time after initiation of detonation but before the detonation wave has reached the piston:

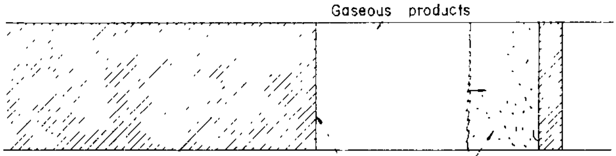


FIG. 1

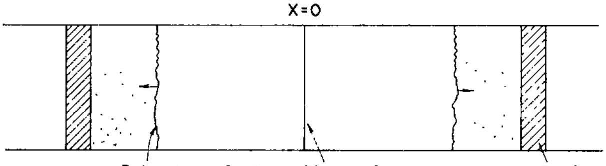
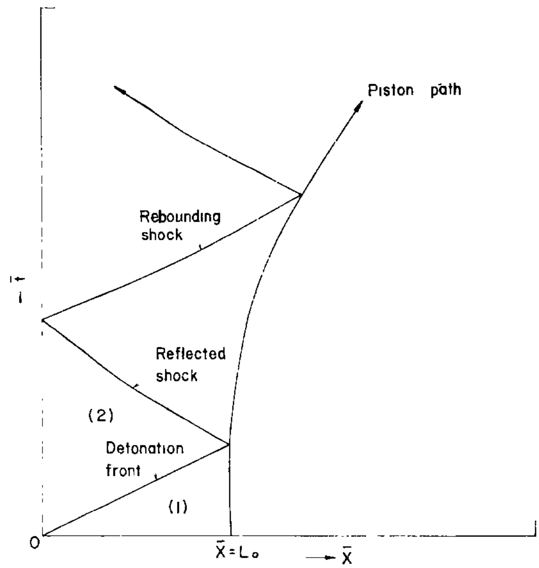


FIG. 2

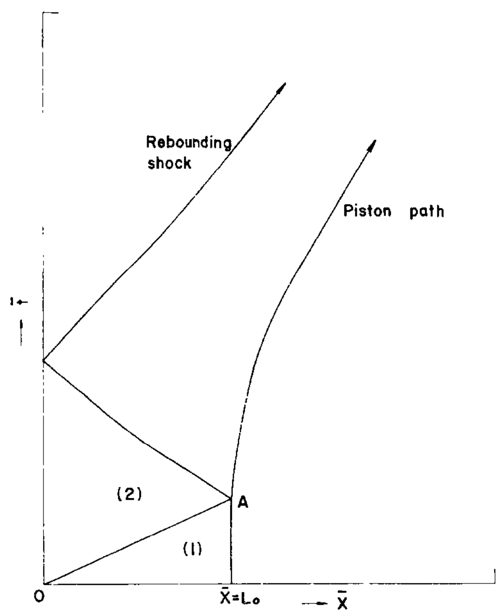
Clearly the two configurations are equivalent and the description of the chain of events is as follows (refering to Fig. 1): a condensed explosive, of length L_0, is detonated at one end ($\bar{x} = 0$). A plane detonation front, of velocity D, will race from this closed end until it reaches the piston. The flow behind this detonation front, before impact upon the piston, has two distinct regions [2]—a quiescent zone extending from the walled—end to half the distance to the front and from this point on to the front a simple centered wave (Taylor wave). When the detonation front overtakes the piston a weak shock is reflected while the piston accelerates smoothly from a zero initial velocity. The reflected shock travels to the left advancing first into the Taylor wave, then it enters the core of quiescence until it reaches the wall and rebounds from it. This rebounding shock wave now moves to the right and might or might not overtake the moving piston, depending on the mass ratio $r = m_e/m_p$ where m_e is the mass of the explosive and m_p is the piston mass.

Figures 3 and 4 are space time diagrams for the two respective cases indicated above. Region (1) is the unreacted explosive, region (2) is the one



Piston path
Rebounding shock
Reflected shock
(2)
Detonation front
(1)
0
$\bar{X} = L_0$
$\rightarrow \bar{X}$

FIG. 3



Rebounding shock
Piston path
(2)
A
(1)
0
$\bar{X}=L_0$
$\rightarrow \bar{X}$

FIG. 4

that includes the quiescent zone and Taylor wave. These two figures should be contrasted to the space time diagram for the case when back expansion of the detonation products is allowed (see Fig. 5 below). The gaseous explosion products are assumed to be polytropic, i.e. their equation of state is of the form

$$\bar{E} = \frac{\bar{p}}{\bar{\rho}(\gamma - 1)} \tag{1.1}$$

where $\bar{E}$ is the specific internal energy. The value $\gamma = 3$ will be used later because it will allow the analysis to be carried to completion. It can be shown [1] that the piston motion is insensitive to changes in γ between 2.5 and 3.5 provided one nondimensiolizes the piston speed properly; at the same time equation (1.1), with $\gamma \simeq 3$, represents quite well gaseous detonation products of organic condensed explosives.

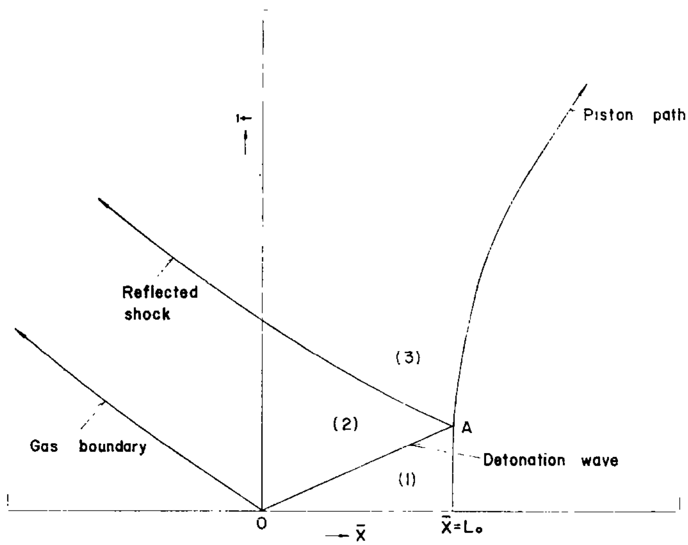


FIG. 5

Previous results, relevant to the present problem, are listed in Section 2. A concise account of the numerical work is given in Section 3. The main results of the present study are described in Section 4: delineation of the flow regimes, explicit formulae for the piston path, reflected shock path, rebounding shock path and the flow fields.

2. Summary of Previous Results

Throughout the calculations the following dimensionless variables are used:

$$x = \frac{\bar{x}}{L_0}, \qquad t = \bar{t} \cdot \frac{D}{L_0},$$

$$U = \frac{\bar{U}}{D}, \qquad C = \frac{\bar{C}}{D}, \qquad \rho = \frac{\bar{\rho}}{\rho_0}, \qquad p = \frac{\bar{p}}{\rho_0 D^2}.$$

Here $\bar{x}$ is the distance, $\bar{t}$ is the time, $\bar{U}$ is the particle velocity, $\bar{C}$ is the sound speed, ρ is the density, ρ_0 is the density of the undetonated explosive and $\bar{p}$ is the pressure.

In region (2) we have an explicit solution given by G. I. Taylor [2]. For the special case of $\gamma = 3$ and with the above nondimensionalization scheme the flow in region 2 is given by:

$$U_2 = \begin{cases} \dfrac{x}{2t} - \dfrac{1}{4} & \dfrac{1}{2} \leqslant \dfrac{x}{t} \leqslant 1 \\[2ex] 0 & 0 \leqslant \dfrac{x}{t} \leqslant \dfrac{1}{2} \end{cases} \tag{2.1}$$

$$C_2 = \begin{cases} \dfrac{x}{2t} + \dfrac{1}{4} & \dfrac{1}{2} \leqslant \dfrac{x}{t} \leqslant 1 \\[2ex] \dfrac{1}{2} & 0 \leqslant \dfrac{x}{t} \leqslant \dfrac{1}{2} \end{cases} \tag{2.2}$$

$$\rho_2 = \frac{16}{9} C_2 \tag{2.3}$$

$$p_2 = \frac{27}{256} \rho_2{}^3 = \frac{1}{4} \left(\frac{3}{4} \rho_2\right)^3 = \frac{16}{27} C_2{}^3. \tag{2.4}$$

These relations satisfy the Chapman-Jouguet conditions on the line OA in the spacetime diagrams, namely:

$$U_{\mathrm{CJ}} = \tfrac{1}{4} \qquad C_{\mathrm{CJ}} = \tfrac{3}{4} \qquad p_{\mathrm{CJ}} = \tfrac{1}{4} \qquad \rho_{\mathrm{CJ}} = \tfrac{4}{3}. \tag{2.5}$$

In the case corresponding to Fig. 5, i.e. when the detonation products are allowed to expand to the left into vacuum, to be designated as the open-end case, the above relations were used by Rostoker and Murray [3] and Aziz *et al.* [1], to obtain expressions for the piston speed. Their analysis is based on the assumption that the reflected shock is weak—therefore the positive characteristics do not change slope upon crossing it, and hence the expression $U + C = x/t$ remains valid on the piston path. The expression they find for the rigid piston speed, U_p, as function of time is:

$$U_p = 1 + \frac{2}{\Omega} - \frac{1+\Omega}{\Omega}\left(1 + \Omega - \frac{\Omega}{t}\right)^{-1/2} - \frac{1}{\Omega}\left(1 + \Omega - \frac{\Omega}{t}\right)^{1/2} \quad (2.6)$$

where Ω is proportional to the mass ratio; $\Omega = 32r/27 = (32/27)(m_e/m_p)$. The terminal velocity, found by letting $t \to \infty$ in Eq. (2.6) is:

$$U_f = \lim_{t\to\infty} U_p = 1 - \frac{2}{\Omega}[\sqrt{1+\Omega} - 1]. \quad (2.7)$$

To determine the flow in region 3, in the "open-end" case, it is necessary to solve the equations of motion subject to the boundary condition that $U = U_p$ on the piston path $x = x_p = \int_1^t U_p(y)\,dy$, and subject to the initial condition that at $t = 1$ (and $x = 1$) $U = 0$, $C = 1$. This has been previously carried out by one of the present authors [4]. The expressions for U and C in region 3 in the "open-end" case were found to be:

$$U_3 = \frac{x}{t} - \frac{1}{t}\left\{\frac{\left(1 + \frac{\Omega}{2}\right)t - \frac{\Omega}{2}x}{(1+\Omega)\,t - \Omega}\right\} \qquad (t \geqslant 1) \quad (2.8)$$

$$C_3 = \frac{1}{t}\left\{\frac{\left(1 + \frac{\Omega}{2}\right)t - \frac{\Omega}{2}x}{(1+\Omega)\,t - \Omega}\right\} \qquad (t \geqslant 1). \quad (2.9)$$

Using the nonconstant part of Eqs. (2.1) and (2.2) in the differential equation for the velocity of rearward looking compression wave the following expression was found for x_S, the reflected shock path:

$$x_S = \frac{1}{2(1+\Omega)}\{4 + 2\Omega - [(1+\Omega)\,t - \Omega]^{1/2} - [(1+\Omega)\,t - \Omega]\}$$

$$(t \geqslant 1). \quad (2.10)$$

The speed of the reflected shock, in the "open-end" case is therefore

$$\frac{dx_S}{dt} = -\frac{1}{2} - \frac{1}{4}\left[\frac{1}{((1+\Omega)\,t - \Omega)^{1/2}}\right] \qquad (t \geqslant 1). \quad (2.11)$$

It should be noted that as $\Omega \to 0$, i.e. an infinitely massive piston, the results (2.8)-(2.11) reduce to the expressions derived by Zeldovich and Stanyukovich [5] for that case.

3. Numerical Results[1]

The piston motion and the flow between the accelerating piston and the back wall were determined by a finite difference procedure. The method used was an explicit Lagrangian scheme using the Von-Neumann and Richtmyer [6], [7] artifical viscosity. The differential and difference equations can be found in any standard text [7] and will not be repeated here. The stability criterion is a modified Courant-Friedrich's-Lewy (C-F-L) one. The equation of motion of the piston was accounted for in the way in which it was done by Aziz *et al.* [1] and the present authors in a previous paper [8], [9]. However the boundary condition on the left-most Lagrangian cell is now that of zero velocity, rather than zero pressure as in the "open-end" case.

Actually there are better finite difference schemes—for example the Lax-Wendroff method [10] and others. Consider Fig. 6 and 7. Figure 6 shows the pressure profiles at various times, computed by the Von-Neumann-Richtmyer method. Figure 7 repeats the calculation using an iterative method [11] which yields smoother profiles. We found out, however, that the piston path was insensitive to the numerical method used. Since we had a ready Von-Neumann-Richtymer code which demanded shorter time to run we used it. All the various finite difference schemes were run with 200, 500 and 1000 Lagrangian cells. Three significant figures in the results are assured with 200 cells.

It was anticipated that there exists a critical mass ratio, r_c , above which the motion of the piston is unaffected by the closing of the "pipe" at $x = 0$.

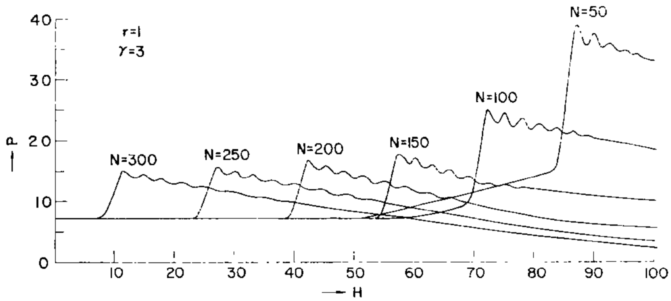


Fig. 6. Pressure profiles at various times computed by the artificial viscosity method.

[1] All the numerical computations were carried on the CDC3400 computer at the Tel-Aviv University Computation center.

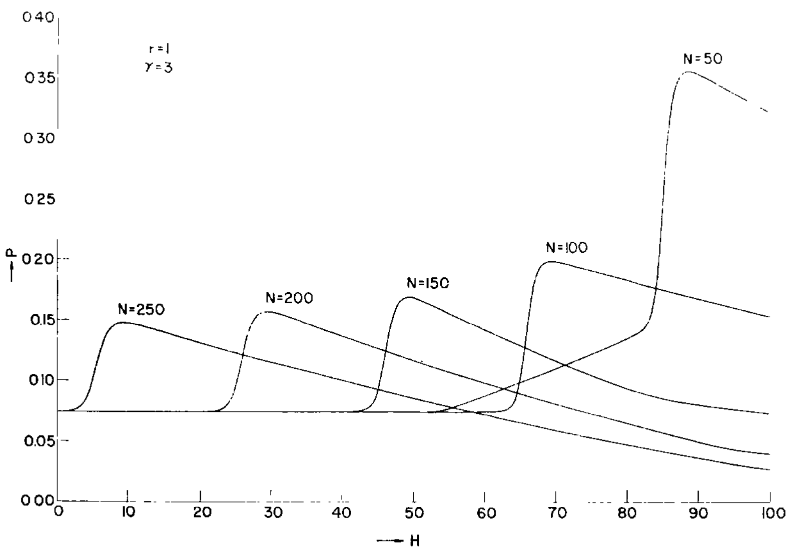


FIG. 7. Pressure profiles at various times computed by an iterative method [11].

Figure 8 shows the dimensionless final piston velocity, $U_f = \bar{U}_f/D$, as function of the mass ratio $r = m_e/m_p$.

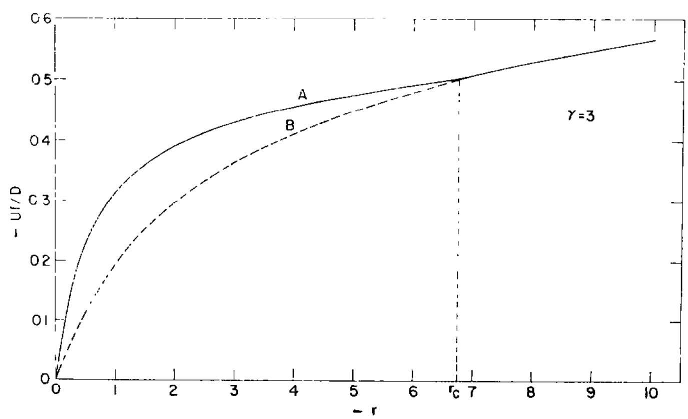


FIG. 8. A—Closed-end, B—Open-end.

The dashed curve is that for the "open-end" case; the solid curve is for the present case of a closed-end pipe. It is seen that above $r = r_c = 6.7$ the two paths coincide. Our first explanation of this phenomenon was that the critical mass ratio, r_c, is determined by the requirement that above $r = r_c$ the piston is so light that the reflected shock after rebounding from the wall at $x = 0$ cannot overtake it again. Indeed Fig. 9 shows in the $x - t$ plane that for $r = 10$ the shock never reaches the piston again, while Fig. 10 shows that

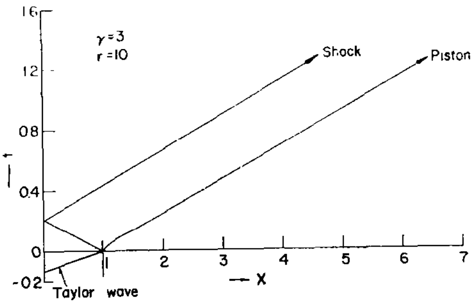


FIG. 9.

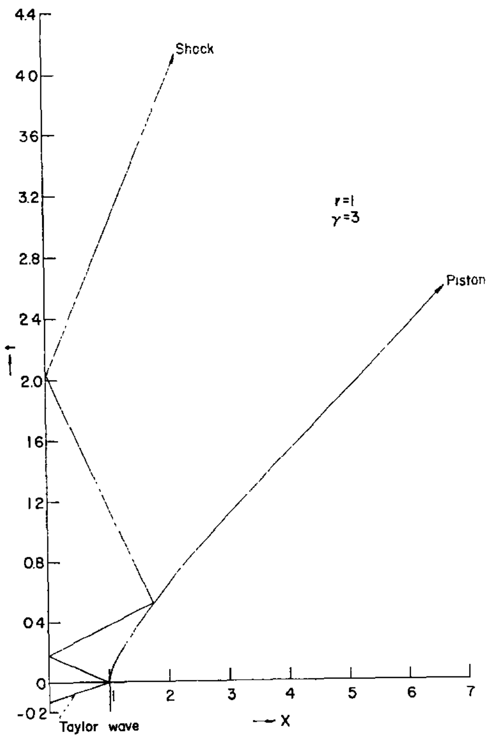


FIG. 10.

when $r = 1$ the shock reaches the piston once again, re-reflects off it, rebounds a second time from the wall $x = 0$ but trails after the piston from that point on. Figure 11 shows that if the piston is heavy enough ($r = 0.5$) then there are two shock reflections off the piston, as might have been expected. (Here we don't count the reflection of the detonation front).

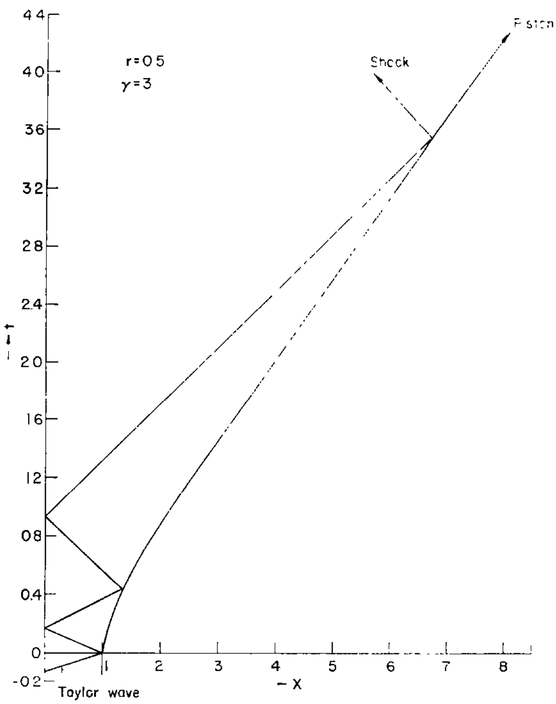


FIG. 11.

Figure 12 shows a typical time history of the piston motion for the case $r = 1$ (one shock reflection off piston) both for the "open-end" and "closed-end" pipe. The time T_c is that required for the rebounding shock to reach the piston. An examination of this graph will show that the piston motion, in the "closed-end" case, is affected considerably before the rebounding shock overtakes it. One must then pose the question: how does the piston sense the fact that the back end is closed when the fastest signal in the fluid, the rebounding shock, has not yet reached it? The table below compares the results for the free and closed end cases. Note that while for $r < 7.1$ it will be shown that the rebounding shock overtakes the piston, yet for $r > 6.7$ the piston motion is already unaffected by the closing of the "pipe" at $x = 0$.

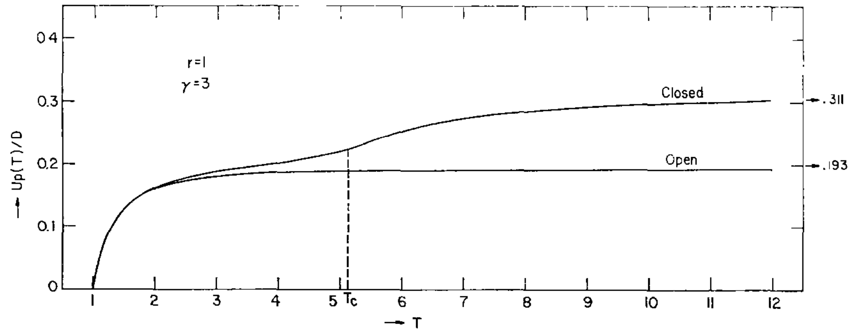


FIG. 12.

r	Numerical Value of U_t for closed end case	U_t for free end
0.50	0.230	0.116
0.60	0.249	0.133
0.65	0.258	0.142
1.00	0.311	0.193
2.00	0.390	0.295
3.00	0.430	0.362
4.00	0.455	0.411
5.00	0.475	0.449
6.00	0.491	0.480
6.50	0.498	0.494
6.80	0.501	0.501
7.00	0.506	0.506
8.00	0.528	0.528
9.00	0.547	0.547
10.00	0.564	0.564
25.00	0.694	0.693

4. Analytic Solution

As in the "open-end" case the analytic treatment here takes advantage of the fact that for $\gamma = 3$ the characteristics are straight lines and that weak shock waves can be represented by compression waves.

It turns out that the line $x = t/2$ (which represents, at least in the beginning, the propagation of the quiescent core) plays an important role in determining the solution.

It turns out that in the analysis it is necessary to distinguish between three regimes:

(a) The mass ratio r is sufficiently small, i.e. the piston is sufficiently heavy, so that the "sonic signal" associated with advance of the stagnant core ($x = t/2$) reaches the piston before the shock rebounding from $x = 0$ overtakes the piston.

(b) An intermediate zone in which the shock wave rebounding from $x = 0$ reaches the piston before any other perturbing signal.

(c) The piston is so light, i.e. r is sufficiently large, that no signal associated with the closure at $x = 0$ reaches the piston. It is anticipated that in this event the piston motion will not differ from that of the "open-end" case.

We shall consider case c first. Its space-time diagram may be represented as below. For $\gamma = 3$, the Riemann invariants are $R^+ = U + C$ and $R^- = U - C$ and the equations of motion may be cast in the following form [8].

$$\frac{\partial R^+}{\partial t} + R^+ \frac{\partial R^+}{\partial x} = 0 \tag{4.1}$$

$$\frac{\partial R^-}{\partial t} + R^- \frac{\partial R^-}{\partial x} = 0 \tag{4.2}$$

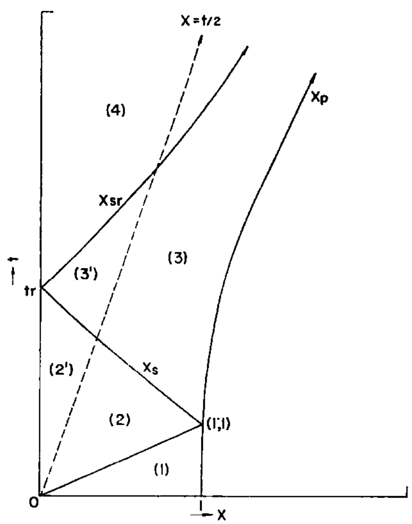


FIG. 13.

from the solution to regions (2) (Taylor wave) and (2′) (quiescent core)—equations (2.1), (2.2)—we determine:

$$R_2^+ = \frac{x}{t}, \qquad R_2^- = -\frac{1}{2}. \tag{4.3}$$

$$R_{2'}^+ = \frac{1}{2}, \qquad R_{2'}^- = -\frac{1}{2}. \tag{4.4}$$

Since the shock wave, x_s, reflecting from the piston to $x = 0$ is assumed weak and is rearward facing, the positive characteristics in region (2) will not change slope upon crossing x_s into region (3). Hence $R_3^+ = x/t$. To determine the solution completely in region (3) we need to know R_3^-. We determine R_3^- as follows:

Consider a point (x, t) in region (3) through this point will pass the two straight characteristics C^+ and C^- whose slopes are given respectively by

$$\frac{1}{R_3^+} = \frac{1}{U_3 + C_3} \quad \text{and} \quad \frac{1}{R_3^-} = \frac{1}{U_3 - C_3}.$$

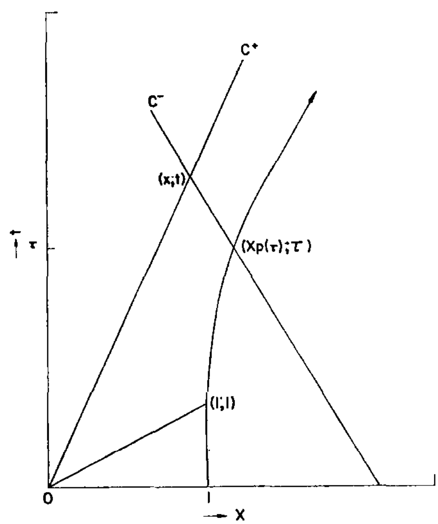


FIG. 14.

We know that $1/R_3^+ = t/x$ and hence $U_3 + C_3 = x/t$. The quantity $1/(U_3 - C_3)$ is invariant along C^- and is equal in particular to $1/(U_p(\tau) - C_p(\tau))$ where the subscript p indicates that the quantities are measured at the piston and τ is the point where C^- intersects x_p. Note that

$\tau = \tau(x, t)$. The same slope of C^- is also given by $(t - \tau)/(x - x_p(\tau))$. Hence we have the equality

$$\frac{x - x_p(\tau)}{t - \tau} = U_p(\tau) - C(\tau). \tag{4.5}$$

As was shown in references (1) and (3) the fact that $U_3 + C_3 = x/t$ suffices to establish U_p and x_p with U_p given by Eq. (2.6). By direct integration with $x_p(1) = 1$ one also finds:

$$x_p(t) = t + \frac{2t}{\Omega}\left[1 - \left(1 + \Omega - \frac{\Omega}{t}\right)^{1/2}\right] \qquad (t \geqslant 1). \tag{4.6}$$

The quantity C_p is found from $C_p = (x_p/t) - U_p$ and thus Eq. (4.5) becomes:

$$\frac{\dfrac{x}{\tau} - \dfrac{x_p(\tau)}{\tau}}{\dfrac{t}{\tau} - 1} = 2U_p(\tau) - \frac{x_p(\tau)}{\tau}. \tag{4.7}$$

If in (4.7) we substitute for $x_p(\tau)$ and $U_p(\tau)$ from (2.6) and (4.6) respectively and multiply by $(t/\tau) - 1$ we obtain after some manipulations:

$$\frac{x}{\tau}\,\xi - \xi - \frac{2}{\Omega}\,\xi + \frac{2}{\Omega}\left(1 + \Omega - \frac{\Omega}{\tau}\right) = \left(\frac{t}{\tau} - 1\right)\left(\xi + \frac{2}{\Omega}\,\xi - \frac{2(1 + \Omega)}{\Omega}\right)$$

where $\xi = (1 + \Omega - (\Omega/\tau))^{1/2}$. After cancellations on both sides, the remaining terms have $1/\tau$ as a common factor which may be cancelled out. The resulting equation for ξ is solved to yield

$$\xi = 2\,\frac{\Omega t + t - \Omega}{\Omega t + 2t - \Omega x}. \tag{4.8}$$

Recall that $x_p(t)$ and $U_p(t)$ are actually functions of $\xi = (1 + \Omega - (\Omega/t))^{1/2}$ only. We can therefore evaluate now $R_3^- = U_p - C_p = 2U_p - (x_p(t)/t)$ by inserting Eq. (4.8) into the expression for U_p and x_p/t, Eqs. (2.6) and (4.6) respectively. After a certain amount of algebraic manipulation one obtains:

$$R_3^- = \frac{1}{t}\left[x - \frac{2t + \Omega(t - x)}{(1 + \Omega)\,t - \Omega}\right] \tag{4.9}$$

from

$$U_3 + C_3 = R_3^+ = \frac{x}{t} \tag{4.10}$$

and

$$U_3 - C_3 = R_3^- = \frac{x}{t} - \frac{1}{t}\left[\frac{2t + \Omega(t - x)}{(1 + \Omega)\,t - \Omega}\right] \tag{4.11}$$

we deduce the expressions for U_3 and C_3 as given by Equations (2.8) and (2.9). The derivation was repeated here since the same technique will be utilized in the other regions as well.

At this stage we calculate the reflected shock trajectory, x_S, seperating region (2) from (3). Since x_S is a weak shock, its velocity dx_S/dt can be approximated as that of a compression wave and be defined by:

$$\frac{dx_S}{dt} = \frac{1}{2}[(U_2 - C_2) + (U_3 - C_3)] \tag{4.12}$$

where the subscripts 2 and 3 refer, respectively, to flow properties in regions (2) and (3). Substituting for U_2, C_2, U_3 and C_3 from Equations (2.1), (2.2), (2.8) and (2.9) the differential equation takes the form

$$\frac{dx_S}{dt} - \left[\frac{1}{2t} + \frac{\Omega}{2t((1+\Omega)t - \Omega)}\right] x_S = -\frac{1}{4} - \frac{1 + \frac{\Omega}{2}}{(1+\Omega)t - \Omega}. \tag{4.13}$$

The solution of (4.13) satisfying the initial condition $x_S(1) = 1$ is given by Equation (2.10). That is the reflected shock trajectory in the "closed-end" case is the same as in the "open-end" case at least between $x = 1$ and the intersection of $x_S(t)$ with $x = t/2$. To the left of that intersection x_S seperates now regions (2′) and (3′) and there is every reason to expect that here the shock trajectory will differ from that found in the "open-end" case. Here too the defining differential equation has the form of Eq. (4.12), i.e.:

$$\frac{dx_S}{dt} = \frac{1}{2}[U_{2'} - C_{2'}) + (U_{3'} - C_{3'})] = \frac{1}{2}[R_{2'}^- + R_{3'}^-]. \tag{4.14}$$

Now, the negative characteristics C^- do not change slope upon crossing the locus of weak singularity, $x = t/2$, and hence $R_{3'}^- = R_3^-$ and, as seen by Equations (4.3) and (4.4) $R_{2'}^- = R_2^- = -\frac{1}{2}$. It follows that the shock trajectory remains the same as given by (4.13) and (2.10). Since $R_{3'}^+ = R_{2'}^+ = \frac{1}{2}$ we can easily find $U_{3'}$ and $C_{3'}$ to be:

$$U_{3'} = \frac{x}{2t} + \frac{1}{4} - \frac{1}{t}\,\frac{\left(1 + \frac{\Omega}{2}\right)t - \frac{\Omega}{2}x}{(1+\Omega)t - \Omega} \tag{4.15}$$

$$C_{3'} = -\frac{x}{2t} + \frac{1}{4} + \frac{1}{t}\,\frac{\left(1 + \frac{\Omega}{2}\right)t - \frac{\Omega}{2}x}{(1+\Omega)t - \Omega}. \tag{4.16}$$

To repeat, the reflected shock trajectory is the same regardless of whether the "pipe" is open or closed at $x = 0$ and is given by

$$x_S = -\frac{t}{2} + \frac{4 + 3\Omega - ((1+\Omega)t - \Omega)^{1/2}}{2(1+\Omega)} \qquad (t \geqslant 1), \quad (0 \leqslant x_S \leqslant 1). \tag{4.17}$$

The time when the shock rebounds from $x = 0$, t_r, is found by putting $x_S = 0$ in Eq. (4.17) and solving for t to get:

$$t_r = \frac{9 + 6\Omega - (8\Omega + 17)^{1/2}}{2(1+\Omega)}. \tag{4.18}$$

Next we should like to find the trajectory of the rebounding shock, x_{Sr}. For this purpose it is necessary to know the flow field in region (4) (see Fig. 13). The invariant connected with the negative characteristic, R_4^-, is the same as R_3^- and $R_{3'}^-$, since this backward facing characteristic is not modified by a forward facing weak shock. On the other hand the slope of the positive characteristic is modified upon crossing from region (3′) to region (4). For proofs of these statements see References (12) and (13). Thus

$$R_4^- = U_4 - C_4 = R_3^- = \frac{x}{t} - \frac{1}{t}\,\frac{(2+\Omega)t - \Omega x}{(1+\Omega)t - \Omega}. \tag{4.19}$$

To find R_4^+ we set

$$R_4^+ = U_4 + C_4 = U_4 + C_4\,|_{x=0} = C_4\,|_{x=0}. \tag{4.20}$$

Consider a positive characteristic in region (4) going through the point (X, t) and crossing the time axis at $(0, \theta)$. We then have, as in the manner of treating region (3),

$$\frac{x}{t-\theta} = C_4(\theta) \tag{4.21}$$

where $C_4(\theta) = C_4(0, \theta)$. But from Eq. (4.19) we have

$$-C_4(\theta) = -\frac{(2+\Omega)}{(1+\Omega)\theta - \Omega}. \tag{4.22}$$

Substituting (4.22) in (4.21) and solving for θ we get

$$\theta(x, t) = \frac{(2+\Omega)t + \Omega x}{(1+\Omega)x + (2+\Omega)}. \tag{4.23}$$

Using this value for θ in (4.22) the expression for $C_4(\theta)$ becomes:

$$R_4{}^+ = U_4 + C_4 = C_4(\theta) = \frac{(1+\Omega)\,x + (2+\Omega)}{(1+\Omega)\,t - \Omega}$$

$$= \frac{x}{t} + \frac{1}{t}\,\frac{(2+\Omega)\,t + \Omega x}{(1+\Omega)\,t - \Omega}\,. \tag{4.24}$$

Adding and subtracting equations (4.19) and (4.24) we obtain the desired flow field expressions

$$U_4 = \frac{(1+\Omega)\,x}{(1+\Omega)\,t - \Omega} \tag{4.25}$$

$$C_4 = \frac{2+\Omega}{(1+\Omega)\,t - \Omega}\,. \tag{4.26}$$

The rebounding shock trajectory is determined by (see ref. 12, p. 159)

$$\frac{dx_{sr}}{dt} = \frac{1}{2}\,[(U_{3'} + C_{3'}) + (U_4 + C_4)] = \frac{1}{2}\,[R_{3'}^+ + R_4{}^+].$$

Using Eq. (4.24) and $R_{3'}^+ = \frac{1}{2}$ the differential equation becomes:

$$\frac{dx_{sr}}{dt} - \frac{\dfrac{1+\Omega}{2}}{(1+\Omega)\,t - \Omega}\,x_{sr} = \frac{1}{4} + \frac{1 + \dfrac{\Omega}{2}}{(1+\Omega)\,t - \Omega} \tag{4.27}$$

with the initial condition that at $x = 0$,

$$t = \frac{6\Omega + 9 - (8\Omega + 17)^{1/2}}{2(1+\Omega)}$$

(see — (4.18)). The resulting trajectory is

$$x_{sr} = \frac{(1+\Omega)\,t - 3\Omega - 4 + ((1+\Omega)\,t - \Omega)^{1/2}}{2(1+\Omega)} \tag{4.28}$$

and the rebounding shock velocity is

$$\dot{x}_{sr} = \frac{dx_{sr}}{dt} = \frac{1}{2} + \frac{1}{4((1+\Omega)\,t - \Omega)^{1/2}}\,. \tag{4.29}$$

Note that $\dot{x}_{rs} = -\,\dot{x}_r$ which one would expect for a reflected compression wave in a stagnant region near a wall.

We now possess enough information to determine r_L, the mass ratio which distinguishes between case (a) and (b). Thus if $r < r_L$ we have case (a); if $r > r_L$ we have either case (b) or (c). We shall later determine r_M, the mass ratio such that if $r > r_M$ then the appropriate regime is case (c).

If $r = r_L$ then $X_p(t)$ (as given by Eq. (4.6)), $x = t/2$ and $x_{sr}(t)$ (as given by Eq. (4.28)) must all meet at one point. The intersection of $X_{sr}(t)$ with $X = t/2$ is given by:

$$t^* = 9\Omega + 16$$
$$x^* = \frac{9}{2}\Omega + 8. \tag{4.30}$$

On the other hand the intersection of $x = t/2$ with

$$x_p = t + \frac{2t}{\Omega}\left[1 - \sqrt{1 + \Omega - \frac{\Omega}{t}}\right]$$

is given by

$$t = \frac{16}{8 - \Omega}$$

$$x = \frac{8}{8 - \Omega} \qquad (\Omega \leqslant 8). \tag{4.31}$$

The expressions (4.30) and (4.31) can be reconciled only for the value of $\Omega = \Omega_L$ satisfying the quadratic equation:

$$9\Omega_L{}^2 - 56\Omega_L - 112 = 0. \tag{4.32}$$

The physically feasible solution is $\Omega_L = 7.816$ or $r_L = 6.595$.

Next we solve for the piston motion and flow field for case (a)—i.e. $r < r_L$. For this value of the mass ratio, or $\Omega < 7.816$ the piston trajectory is reached by the lime $x = t/2$ at the time $t = 16/(8 - \Omega)$. This is the first instant at which the piston motion is affected by a signal which is due to the wall at $x = 0$. The effect is to accelerate the piston, compared to the case of an "open-end" pipe, since the signal line $x = t/2$ is associated with the quiescent core which carries higher pressures than the Taylor wave. The space-time diagram is now as follows:

Between the times $t = 1$ and $t = 16/(8 - \Omega)$ the piston motion is given, as for the "open-end" case, by Eq. (4.6). Between $t = 16/(8 - \Omega)$ and t_5 the piston trajectory is determined by the system:

$$\frac{dU_p}{dt} = \frac{r}{4}p$$
$$p = \frac{64}{27}C_p{}^3$$
$$U_p + C_p = \tfrac{1}{2}. \tag{4.33}$$

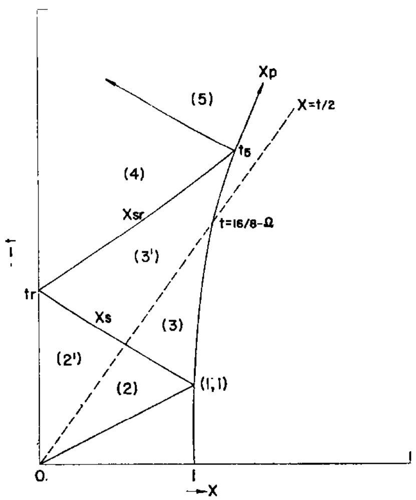


FIG. 15.

The last equality in (4.33) is due to the fact that under our assumption of weak shocks $R_{3'}^{+} = R_3^{+}$. The system (4.33) is equivalent to

$$\frac{dC_p}{dt} = -\frac{\Omega}{2} C_p^3 \tag{4.34}$$

with the initial condition being

$$C_p = \frac{8-\Omega}{4\Omega+16} \qquad \left(t = \frac{16}{8-\Omega}\right). \tag{4.35}$$

The initial condition is obtained from $C_p = \frac{1}{2} - U_p$ with U_p being given by Eq. (2.6) with $t = 16/(8 - \Omega)$. The solution to (4.34), (4.35) is

$$U_p(t) = \frac{1}{2} - \left\{\Omega t + 32\frac{\Omega^2+8}{(\Omega-8)^2}\right\}^{-1/2}. \tag{4.36}$$

If we integrate (4.36) so that for $t = 16/(8 - \Omega)$, $x_p = 8/(8 - \Omega)$ we'll get the piston motion for all $\Omega < 7.816$ between the time $t = 16/(8 - \Omega)$ and the time $t = t_5$ when the piston is overtaken by the rebounding shock wave. The resulting trajectory is

$$x_p = \frac{t}{2} - \frac{8}{\Omega}\left(\frac{4+\Omega}{8-\Omega}\right)\cdot\left\{\left[1+\Omega\left(\frac{8-\Omega}{4+\Omega}\right)^2\left(\frac{t}{16}-\frac{1}{8-\Omega}\right)\right]^{1/2}-1\right\}$$

$$\left(\frac{16}{8-\Omega} < t < t_5\right). \tag{4.37}$$

The time t_5 is found by equating Eq. (4.37) with the trajectory of x_{3r} as given by (4.28). After some tedious arithmetic one obtains:

$$t_5 = \frac{1+\Omega}{2} B^2 + A - B\left[\left(\frac{1+\Omega}{2}\right)^2 B^2 + A + \Omega(A-1)\right]^{1/2} \tag{4.38}$$

where:

$$A = \frac{256(1+\Omega) + 32(4+\Omega)(3\Omega+4)}{(8-\Omega)(16+15\Omega)} + \frac{\Omega(9\Omega^2+23\Omega+16)}{(1+\Omega)(16+15\Omega)}$$

$$B = \frac{32(1+\Omega)(4+\Omega) + 2\Omega(8-\Omega)(3\Omega+4)}{(8-\Omega)(1+\Omega)(16+15\Omega)}.$$

Thus, a typical value is $t_5 = 5.307$ for $\Omega = 1$.

After $t = t_5$ the piston motion is determined by the system

$$\frac{dU_p}{dt} = \frac{r}{4} p$$

$$p = \frac{64}{27} C_p^3$$

$$U_p + C_p = R_5^+ = R_4^+ = \frac{(1+\Omega)x + (\Omega+2)}{(1+\Omega)(t-\Omega)}.$$

Let

$$\xi = (1+\Omega)x + (2+\Omega)$$

$$\tau = (1+\Omega)t - \Omega$$

$$\alpha = C\tau = \xi - U_p\tau.$$

We can then cast (4.39) in the form:

$$\frac{d\alpha}{d\tau} = -\frac{\Omega}{2(1+\Omega)} \frac{\alpha^3}{\tau^2}. \tag{4.40}$$

Where the initial values at $t = t_5$ are:

$$\tau_5 = (1+\Omega)t_5 - \Omega$$

$$x_5 = \frac{t_5}{2} - \frac{8}{\Omega}\left(\frac{4+\Omega}{8-\Omega}\right)\left\{\left[1 + \Omega\left(\frac{8-\Omega}{4+\Omega}\right)^2\left(\frac{t_5}{16} - \frac{1}{8-\Omega}\right)\right]^{1/2} - 1\right\}$$

$$\xi_5 = (1+\Omega)x_5 + (2+\Omega)$$

$$U_5 = U_p(t_5) = \frac{1}{2} - \left\{32\left[\frac{\Omega^2+8}{(\Omega-8)^2}\right] + \Omega t_5\right\}^{-1/2}$$

$$\alpha_5 = \xi_5 - U_5\tau_5. \tag{4.41}$$

The solution of Eq. (4.40) is

$$\alpha = \frac{\alpha_5}{\left(1 + \frac{\alpha_5^{\,2}}{2\tau_5} - \frac{\alpha_5^{\,2}}{2\tau}\right)^{1/2}} . \tag{4.42}$$

Now from $U_p\tau + \alpha = \xi$ it follows upon differentiation with respect to τ that

$$\frac{dU_p}{d\tau} = -\frac{1}{\tau}\frac{d\alpha}{d\tau}$$

and hence

$$\int_{U_5}^{U_p} dU_p = -\int_{\alpha_5}^{\alpha} \frac{d\alpha}{\tau(\alpha)} .$$

Using Eq. (4.42) to substitute for $\tau(\alpha)$ and carrying out the integration we get

$$U_p = U_5 + (\alpha_5 - \alpha)\left(\frac{1}{\tau_5} + \frac{1+\Omega}{\Omega\alpha_5^{\,2}} - \frac{1+\Omega}{\Omega\alpha\alpha_5}\right) . \tag{4.43}$$

The asymptotic value of $U_p(t)$ is found by letting $\tau \to \infty$ in (4.42) and (4.43):

$$U_f = \lim_{t\to\infty} U_p(t)$$
$$= U_5 + \left(1 - \left(\frac{2\tau_5}{2\tau_5 + \alpha_5^{\,2}}\right)^{1/2}\right)\left\{\frac{\alpha_5}{\tau_5} + \frac{1+\Omega}{\Omega\alpha_5}\left(1 - \left(1 + \frac{\alpha_5^{\,2}}{2\tau_5}\right)^{1/2}\right)\right\} . \tag{4.44}$$

Using this result in conjunction with the initial values from Eq. (4.41) we computed final piston velocities for various values of $r < r_L$ and compared them with the values obtained by numerical integration.

The table below, citing some of the values, shows that the maximum difference between of the analytical and numerical solution of less than 3%, occurs for the lowest mass ratio.

r	Ω	U_f Analytical; eq. (4.44)	U_f Numerical solution	U_f Open end at $x = 0$
1.0	1.1852	.302	.311	.193
2.0	2.3704	.380	.390	.295
3.0	3.5556	.421	.430	.362
4.0	4.7407	.447	.455	.411
5.0	5.9259	.468	.475	.449
6.0	7.1111	.487	.491	.480
6.5	7.7037	.496	.498	.494

It is interesting to note that the effect of rerefaction wave due to the shock rebounding from the piston at $t = t_5$ is to decelerate the piston. This can be seen from the fact that if the $t_5 \to \infty$ then $U_5 \to \frac{1}{2}$ for all $r < 6.595$ which is larger than the values actually obtained.

Having obtain the solution for $r < r_L = 6.595$ and $r > r_M$ there remains to determine the solution in the intermediate range of $r_L < r < r_M$ and also find the value of r_M. For this range of the mass ratio the shock wave rebounding from $x = 0$ reaches the piston before any other signal. The situation is shown below in Fig. 16:

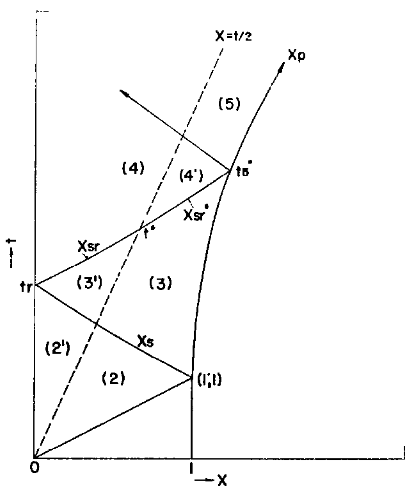


FIG. 16.

From Eq. (4.30) we find that the rebounding shock wave overtakes the "signal" $x = t/2$ at the time $t = t^* = 9\Omega + 16$. From that instant on it has in front of it the region (4′) in which the negative invariant $R_{4'}^-$ is the same as R_3^- since the forward—facing weak shock cannot interact with a negative characteristic. Thus

$$R_{4'}^- = \frac{x}{t} - \frac{1}{t}\,\frac{(2+\Omega)\,t - \Omega x}{(1+\Omega)\,t - \Omega}\,. \tag{4.45}$$

We now employ the same procedure as before in order to determine $R_{4'}^+$ except that the point τ is *not* the intersection of the positive characteristic with $x = 0$ but with $x = t/2$. It is found in this manner that

$$R_{4'}^+ = \frac{x}{t} + \frac{1}{t}\,\frac{(2+\Omega)\,t - \Omega x}{(1+\Omega)\,t - \Omega}\,. \tag{4.46}$$

Comparing (4.45) and (4.46) with (4.19) and (4.24) respectively we see that $R_{4'}^- = R_4^-$ and $R_{4'}^+ = R_4^+$. Thus regions (4′) and (4) are actually one single region and the "signal line" $x = t/2$ plays no part in the analysis beyond $t = t^* = 9\Omega + 16$; the "memory" of the quiescent core has been erased.

Even though region (4′) is identical to (4) it is still necessary to solve for the rebounding shock motion beyond $t = t^* = 9\Omega + 16$; this is because $R_3^+ \neq R_{3'}^+$. The differential equation for the shock trajectory is

$$\frac{dx_{\mathrm{sr}}^*}{dt} = \frac{1}{2}\,[R_{4'}^+ + R_3^+] \tag{4.47}$$

with the initial condition $x_{\mathrm{sr}}^*(t^*) = \frac{9}{2}\,\Omega + 8$. The solution is

$$x_{\mathrm{sr}}^* = t \cdot \left[\frac{\sqrt{9\Omega + 16}}{2\Omega}\sqrt{1 + \Omega - \frac{\Omega}{t}} - \left(1 + \frac{2}{\Omega}\right)\right]. \tag{4.48}$$

In order to find the time, $t_5{}^*$, when the rebounding shock overtakes the piston we equate x_{sr}^* from Eq. (4.48) with x_{p} given by Eq. (4.6). Solving the resulting expression for $t_5{}^*$ yields:

$$t_5{}^* = \frac{\Omega(9\Omega + 32 + 8(9\Omega + 16)^{1/2}}{8(1 + \Omega)\,(9\Omega + 16)^{1/2} - (7\Omega^2 + 23\Omega + 32)}\,. \tag{4.49}$$

From (4.49) we can determine r_{M} since there is a maximum value of Ω, Ω_{M}, for which the denominator of Eq. (4.49) is nonnegative. For $\Omega > \Omega_{\mathrm{M}}$ the rebounding shock cannot overtake the piston. We find that the equality

$$8(1 + \Omega)\sqrt{9\Omega + 16} = 7\Omega^2 + 23\Omega + 32$$

is satisfied by a certain value of Ω, $8 < \Omega_{\mathrm{M}} < 9$, which solves the cubic algebraic equation

$$49\Omega_{\mathrm{M}}{}^3 - 254\Omega_{\mathrm{M}}{}^2 - 1199\Omega_{\mathrm{M}} - 1152 = 0. \tag{4.50}$$

The proper root of (4.50) was found, using Newton–Raphson iterations to be:

$$\Omega_{\mathrm{M}} = 8.42098 \quad \text{or, equivalently:} \quad r_{\mathrm{M}} = 7.1043 \tag{4.51}$$

Thus the intermediate region, case (b), covers the range $6.595 < r < 7.104$. In this range of the mass ratio we have to solve for the piston motion starting with the time $t = t^*$ given by Eq. (4.49). Since $R_{4'}^+ = R_4^+ = R_5^+$, the differential equation system to be solved is identical to (4.39) except that now the initial conditions are to be specified at $t = t_5^*$. The final piston velocity is found to be

$$U_f = U_5^* + (W - 1)\left[\frac{1}{(1+\Omega)\,t^* - \Omega} + \frac{1+\Omega}{\Omega\alpha_5^*} - \frac{1+\Omega}{\Omega\alpha_5 W}\right] \qquad (4.52)$$

where

$$U_5^* = U_p(t_5^*) = 1 + \frac{2}{\Omega} - \frac{(1+\Omega)\,(9\Omega + 32 + 8(9\Omega+16)^{1/2})^{1/2}}{4\Omega(\Omega+2)}$$

$$- \frac{4(\Omega+2)}{\Omega(9\Omega + 32 + 8(9\Omega+16)^{1/2})^{1/2}}$$

$$W = \lim_{t\to\infty}\frac{\alpha}{\alpha^*} = 1 + \left\{\frac{\Omega\alpha_5^*}{(1+\Omega)\,[(1+\Omega)\,t_5^* - \Omega]}\right\}^{-1/2}$$

$$\alpha_5^* = [(1+\Omega)\,x_5^* + (2+\Omega)] - U_5^*[(1+\Omega)\,t^* - \Omega]$$

$$x_5^* = x(t_5^*) = t_5^*\left[1 + \frac{2}{\Omega} - \frac{8(2+\Omega)}{\Omega(9\Omega + 32 + 8(9\Omega+16)^{1/2})^{1/2}}\right].$$

A table similar to that given for $r < r_L$ was constructed here, using Eq. (4.52), for $r_L < r < r_M$. Comparing, in this table, the results for the present "closed-end pipe" with the results for the "free-end" case, we see that indeed at $r = 7.1$ they are identical and that the biggest difference, at $r = 6.6$, is less than $\frac{1}{2}\%$. Thus for practical purposes it is only in the region of case (a), $r < r_L$, that there is a marked departure from the case treated by Aziz *et al.* This is the range where most of the acceleration is due to the advancing quiescent core which overtakes the piston. In the overall picture the shock rebounding from $x = 0$ plays a relatively minor role.

r	Ω	U_f Analytical; eq. (4.52)	U_f Numerical solution	U_f Open end at $x = 0$
6.6	7.8222	.4978	.4980	.4960
6.7	7.9407	.4998	.4994	.4987
6.75	8.0000	.5008	.5002	.5000
6.8	8.0593	.5018	.5009	.5010
6.9	8.1778	.5039	.5041	.5037
7.0	8.2963	.5061	.5060	.5060
7.1	8.4148	.5084	.5084	.5084

There is one point which we have glossed over so far. This is the question of the piston motion for very small r, i.e. massive pistons. When $r < .62$ it was found numerically that the rebounding shock wave can reach the piston twice and as $r \rightarrow 0$ the number of shock reflections from the piston incresses without bound. While the technique employed in the present paper can, in principle, be extended to any number of rebounds and reflections the complexity of the analysis will outweigh the advantages inherent in an analytic solution. It is possible, however, to obtain a good approximation to the asymptotic piston speed using what might be called an adiabatic assumption concerning the flow field.

We shall assume that, for small r, the velocity field after some time $t > T$ may be represented by

$$U(x, t) = \frac{U_{\mathrm{P}}(t)}{x_{\mathrm{P}}(t)} \cdot x = A(t) \cdot x \qquad (t > T) \tag{4.53}$$

while it still satisfies the boundary conditions

$$\begin{aligned} U(0, t) &= 0 \\ U(x_{\mathrm{P}}, t) &= U_{\mathrm{P}}(t). \end{aligned} \tag{4.54}$$

It is implicit in the above assumption that T is large enough so that the reflecting shocks have decayed and thus $U(x, t)$ is a smooth function. The equations governing the flow field (for $\gamma = 3$) are:

$$C_t + UC_x + CU_x = 0 \tag{4.55}$$

$$U_t + UU_x + CC_x = 0. \tag{4.56}$$

If we substitute Eq. (4.53) in (4.56) and carry out the integration with respect to x we get immediately:

$$C^2(x, t) = -(A_t + A^2)\, x^2 + F(t) \tag{4.57}$$

where $F(t)$ is an unknown function of t. Next multiply Eq. (4.55) by C and substitute for U, C^2 and its derivatives from (4.53) and (4.57) to get:

$$F_t + 2AF - (A_{tt} + 6AA_t + 4A^3)\, x^2 = 0. \tag{4.58}$$

Equation (4.58) can be satisfied by the pair of differential equations:

$$A_{tt} + 6AA_t + 4A^3 = 0 \tag{4.59}$$

$$F_t + 2AF = 0. \tag{4.60}$$

A solution of (4.59) that satisfies $\lim_{t \to \infty} A(t) = 0$ is $A(t) = 1/t$. It follows from it that $F = b^2/t^2$ where b is a constant to be determined shortly.

Substituting $A = 1/t$ and $F = b^2/t^2$ in Eqs. (4.53) and (4.57) gives us the flow field for $t > T$:

$$U(x, t) = \frac{x}{t}$$

$$C(x, t) = \frac{b}{t} \qquad (t > T). \tag{4.61}$$

We also require that the total energy in the system be conserved at all times. In terms of the dimensional quantities this statement is:

$$\int_0^{\bar{x}_p} \frac{1}{2} \bar{\rho}\bar{u}^2 \, d\bar{x} + \int_0^{\bar{x}_p} \frac{\bar{p}}{(\gamma - 1)} \, d\bar{x} + \frac{1}{2} m_p \bar{u}_p{}^2 = \frac{\rho_0 L_0 D^2}{2(\gamma^2 - 1)} \tag{4.62}$$

where ρ_0 and L_0 are, as before, the density and length of the undetonated explosive. The term on the right hand side is, then, the initial chemical energy (per unit cross-sectional area) stored in the unreacted material. Using the dimensionless scheme of Section 2, substituting for U and C from Eq. (4.61), and carrying out the indicated integrations we get:

$$\frac{8}{27} b \left(\frac{x_p}{t}\right)^3 + \frac{8}{27} \cdot \frac{b^3}{t^2} \left(\frac{x_p}{t}\right) + \frac{1}{2} \frac{U_p{}^2}{r} = \frac{1}{16} \qquad (t > T). \tag{4.63}$$

We are interested in the asymptotic value of U_p; $U_f = \lim_{t\to\infty} U_p$, so that in Equation (4.63) we may set $\lim_{t\to\infty}(x_p(t)/t) = U_f$. Taking this limit we have

$$\frac{1}{2r} U_f{}^2 + \frac{8b}{27} U_f{}^3 - \frac{1}{16} = \lim_{t\to\infty} \left(\frac{8b^3}{27} \cdot \frac{U_f}{t^2}\right) = 0$$

or, equivalently

$$\frac{16r}{27} bU_f{}^3 + U_f{}^2 - \frac{r}{8} = 0 \tag{4.64}$$

where we still have not determined the constant b. It is clear that the solution must satisfy:

$$0 < U_f < \sqrt{\frac{r}{8}}. \tag{4.65}$$

Notice that the value $\sqrt{r/8}$, which corresponds to the assumption that *all* the initial chemical energy has been transfered to the piston, gives for $r = \frac{1}{2}$ the numerical value of $U_f = 0.25$ as compared with $U_f = 0.239$ obtained by the complete numerical integrations of the problem.

To get explicit solution to Eq. (4.64) we fix b in such a manner that for some mass ratio ($r > 0.62$) the solution of (4.64) will agree with that given by the previous complete analysis. To be specific we required that at $r = 1$ Eq. (4.64) yield the value $U_f = 0.3$ obtained from Eq. (4.44). Eq. (4.64) thus becomes:

$$\frac{280}{27} U_f^3 + \frac{8}{r} U_f^2 - 1 = 0. \tag{4.66}$$

The root of interest to us is, as we saw, between 0 and $\sqrt{r/8}$ and it was computed, using Newton's method, for various values of $r < 1$. As can be seen from the accompanying table the maximum error, at $r = \frac{1}{2}$, is 1.3%.

r	$\sqrt{\frac{r}{8}}$	U_f Numerical Solution	U_f Equation (4.66)
0.50	0.2500	0.2301	0.2330
0.25	0.1767	0.1703	0.1721
0.10	0.1118	0.1100	0.1110

5. Summary

The problem considered was that of determining the flow field between an infinitely massive wall at $x = 0$ and a rigid piston being accelerated by the products of detonation of a condensed explosive which originally filled the space $0 \leqslant x \leqslant x_p(0) = 1$. The piston path is also determined. It was shown that this can be done with analytical tools when the combustion products are assumed to have a polytropic equation of state with $\gamma = 3$. The analyticial results were compared with those obtained from a finite-difference numerical integration of the equations of motion. Close agreement was found between the two sets of calculations. The use of the theoretical approach removed an apparent anomaly connected with the numerical results. It was found that there are four main regines depending on the mass ratio:

(1) $r < 0.62$; heavy piston with multiple shock reflections. An adiabatic assumption allow to compute the asymptotic piston speed and flow fields from equations (4.64) and (4.61) respectively.

(2) $0.62 < r < r_L = 6.595$; quiescent core reaches piston before the reflected shock wave—see Fig. 13. There is a marked departure from the results of an open end at $x = 0$.

(3) $r_L < r < r_M = 7.104$; reflected shock wave overtakes piston before quiescent core (which is obliterated when intersection with shock occurs; see Fig. 11). The final piston speed deviations from the open-end case are slight and not easily detected by numerical methods.

(4) $r_M < r$; no signal indicating a closed end at $x = 0$ reaches piston. Final piston speed same as in the open end case.

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